Using Game Theory to Get a Date

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Every day, Ann arrives home from work at some (random) time A between 4:00 and 5:00, and goes out to eat at 5:00.

Bill and Carl would both like to take Ann to dinner, but both of their cell phone batteries are nearly dead, with enough power for only one call.

If Bill calls at time B and Carl calls at time C, Bill gets the date if either $A < B < C$ or $C < A < B$. (If both B and C are less than A, everyone dines alone.)

What calling strategy should Bill use to maximize his chance of getting a date (regardless of Carl's choice of calling time)?

adapted from *Statistical Decision Theory and Bayesian Analysis*,
James O. Berger (Springer, 1985)

Assumptions and comments:

- Ann's arrival time A is uniformly distributed from $t = 0$ (4:00) to $t = 1$ (5:00).

  ... but we will consider variations later.

- We seek a *randomized* strategy; that is, each day, Bill chooses a random calling time.

  If there were some pattern to Bill's calling time, Carl might discern that pattern and get an advantage.

- This is a *zero-sum game,* meaning that if both "players" (Bill and Carl) play optimally, neither can gain an advantage.

  So we aren't trying to *win;* instead, we are trying to *tie.*

Let

$$p(c) = P(\text{Carl gets the date} \,|\, C = c), \text{ and}$$
$$q(c) = P(\text{Bill gets the date} \,|\, C = c).$$

We seek an *equalizer strategy* for Bill's calling time B. That is, we want a function $F(t) = P(B \leq t)$ chosen in such a way that

$$p(c) = q(c) \text{ for all choices of } c.$$

(As we shall see, this is not entirely true. If Carl chooses to call too early, Bill will have a slight edge.)

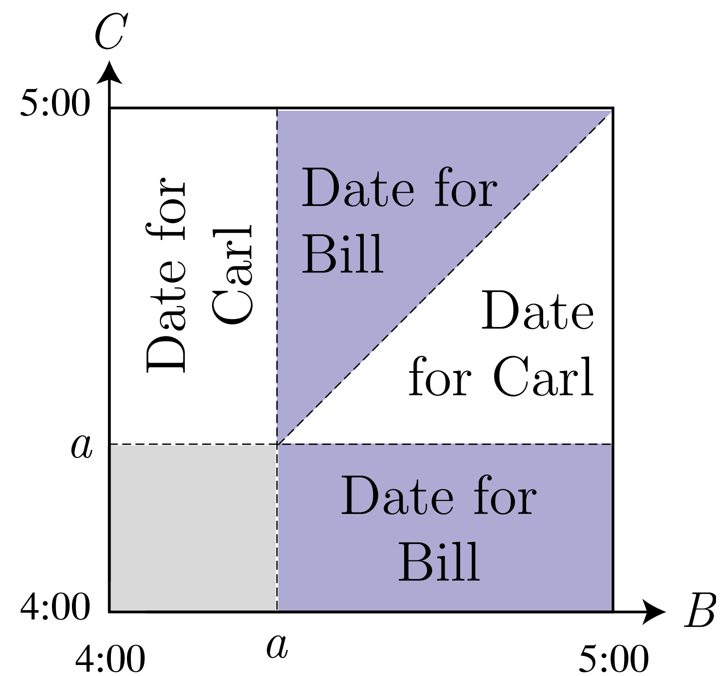
What does this equalizer strategy F look like?

- $F(1) = 1$, because for any reasonable strategy, we should call no later than 5:00.
- We assume F is continuous and differentiable (as $t \to 1^-$).
- Let μ be Bill's average calling time:

$$\mu = \int_0^1 t\,F'(t)\,dt = \int_0^1 (1 - F(t))\,dt$$

(The latter formula can be proved using integration by parts.)

Let's consider the outcomes for a fixed value of a (Ann's calling time).



Both Bill and Carl called too early in the gray area:

$$B < a \text{ and } C < a.$$

Bill gets the date in the two purple regions:

$$C < a < B \text{ or } a < B < C.$$

Carl gets the date in the two white regions:

$$B < a < C \text{ or } a < C < B.$$

For a fixed calling time c for Carl, let

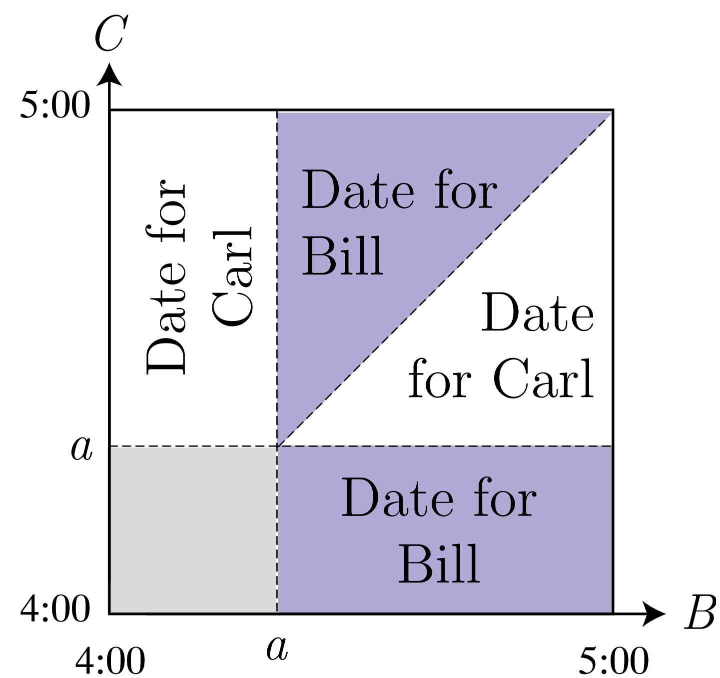
$$p_a(c) = P(\text{Carl gets the date} \mid A = a, C = c) \quad \text{and}$$

$$q_a(c) = P(\text{Bill gets the date} \mid A = a, C = c)$$

Then

$$p_a(c) = \begin{cases} 1 - F(c) + F(a) & \text{if } a < c \\ 0 & \text{if } a > c \end{cases}$$

$$q_a(c) = \begin{cases} F(c) - F(a) & \text{if } a < c \\ 1 - F(a) & \text{if } a > c \end{cases}$$



Integrating over the possible values of A, we find

$$\begin{aligned}
p(c) &= \int_0^1 p_a(c)\,da \\
&= \int_0^c (1 - F(c) + F(a))\,da \\
&= c - c\,F(c) + \int_0^c F(a)\,da \\
q(c) &= \int_0^1 q_a(c)\,da \\
&= \int_0^c F(c)\,da + \int_c^1 1\,da - \int_0^1 F(a)\,da \\
&= 1 - c + c\,F(c) - \int_0^1 F(a)\,da
\end{aligned}$$

The equalizer strategy has $p(c) = q(c)$, so we need

$$2c - 2c\,F(c) + \int_0^c F(a)\,da = 1 - \int_0^1 F(a)\,da$$

Differentiating both sides with respect to c gives

$$2 - 2F(c) - 2cF'(c) + F(c) = 0$$

or equivalently

$$F(c) + 2cF'(c) = 2$$

With the initial value $F(1) = 1$, the solution is

$$F(t) = 2 - \frac{1}{\sqrt{t}} \quad \text{for } \tfrac{1}{4} \le t \le 1$$

(See the end of this document for details.)

So, under the equalizer strategy, Bill should not call before 4:15. His average call time is

$$\mu = \int_0^1 (1 - F(t))dt = \int_0^1 tF'(t)\,dt = \frac{1}{2}$$

If $c > 1/4$, then

$$p(c) = q(c) = \frac{1}{2} + c - \sqrt{c}.$$

However, if Carl calls before 4:15—at time $c < 1/4$—then

$$p(c) = c \quad \text{and} \quad q(c) = \frac{1}{2} - c$$

(The equalizer DE does not apply for times before 4:15, because F is not differentiable at $t = 1/4$.)

Notice that for $c > 1/4$, $p(c)$ and $q(c)$ are strictly increasing, and so are maximized at $c = 1$, where $p(1) = c(1) = 1/2$. Therefore, if Bill follows the equalizer strategy, while Carl always waits until the last minute to place his call, Ann will go out with each suitor half the time.

If both Bill and Carl follow the equalizer strategy, then they each get a date with probability

$$\int_0^1 p(c)F'(c)\,dc = 1 - \ln 2 \doteq 0.307$$

What if Ann's arrival time is not uniform?

In particular, let $H(t) = P(A \leq t)$ for $0 \leq t \leq 1$.

The formulas for $p_a(c)$ and $q_a(c)$ are as before; integrating over the possible values of A gives

$$p(c) = H(c)(1 - F(c)) + \int_0^c F(a)H'(a)\,da$$

$$q(c) = 1 - H(c)(1 - F(c)) - \int_0^1 F(a)H'(a)\,da$$

For the equalizer strategy, we again set $p(c) = q(c)$ and differentiate, which leads to the differential equation

$$H'(c)F(c) + 2H(c)F'(c) = 2H'(c)$$

(with initial value $F(1) = 1$). The solution is

$$F(t) = 2 - \frac{1}{\sqrt{H(t)}} \quad \text{for } H^{-1}(\tfrac{1}{4}) \leq t \leq 1$$

Of course, this is consistent with our previous result for a uniform arrival time, for which $H(t) = t$.

Again we see that Bill should not call before there is a 25% chance that Ann has arrived at home. If Carl calls after that time, then

$$p(c) = q(c) = \frac{1}{2} + H(c) - \sqrt{H(c)}$$

If Carl calls before that time, then

$$p(c) = H(c) \quad \text{and} \quad q(c) = \frac{1}{2} - H(c)$$

Once again, $p(c)$ and $q(c)$ are strictly increasing for $c > H^{-1}(1/4)$.

Also as before, if both Bill and Carl follow the equalizer strategy, then they each get a date with probability $1 - \ln 2 \doteq 0.307$.

One more variation

Suppose that Ann has a third suitor, David.

If Bill and Carl don't know that David is calling, will the two-person equalizer strategy still work?

No . . . in fact, it is not hard for David to choose a strategy which gives him a fairly substantial advantage over Bill and Carl.

This raises the question — is there an equalizer solution to the three-suitor problem?

Appendix: Solving the DE

For the differential equation $F(t) + 2tF'(t) = 2$, divide both sides by $2\sqrt{t}$, yielding

$$\frac{F(t)}{2\sqrt{t}} + F'(t)\sqrt{t} = \frac{1}{\sqrt{t}}$$

This is an exact DE; the left side is the derivative of $F(t)\sqrt{t}$, so antidifferentiating both sides leads to

$$F(t)\sqrt{t} = 2\sqrt{t} + C$$

which, together with the initial condition $F(1) = 1$ leads to the solution

$$F(t) = 2 - \frac{1}{\sqrt{t}} \quad \text{for } \tfrac{1}{4} \le t \le 1$$

For the more general case where Ann's arrival time is not uniform, the DE is

$$H'(t)F(t) + 2H(t)F'(t) = 2H'(t)$$

which is made exact by dividing both sides by $2\sqrt{H(t)}$:

$$\frac{H'(t)}{2\sqrt{H(t)}}F(t) + F'(t)\sqrt{H(t)} = \frac{H'(t)}{\sqrt{H(t)}}$$

Antidifferentiating both sides gives

$$F(t)\sqrt{H(t)} = 2\sqrt{H(t)} + C$$

so the optimal strategy is

$$F(t) = 2 - \frac{1}{\sqrt{H(t)}} \quad \text{for } H^{-1}(\tfrac{1}{4}) \le t \le 1$$